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PSYCHOPHYSICAL LAWS IN RISK THEORY *

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In this article specific hypotheses on the shape of a rational agent's risk preference function are derived from psychophysical laws. Weber's law is used to establish the hypothesis of constant relative risk aversion for a myopic expected-utility maximizer. Weber's law, Fechner's law and a modified version of Koopmans' preference functional are shown to generate a family of multiperiod preference functionals which are either of an additive logarithmic or a multiplicative Cobb-Douglas type. This family has very appealing implications in a world of stochastic constant returns to scale. For the actual decision the multiperiod optimizer exhibits constant relative risk aversion as does the myopic optimizer. However, with the passage of time, the degree of this risk aversion, in general, moves towards unity. Moreover, it is worth noting that the agent neither has to make the consumption decision simultaneously with the selection of an optimal risk project nor needs any information about the future except his or her own preferences.

1. Introduction

Suppose *homo oeconomicus* has to choose one out of a set of mutually exclusive probability distributions of end-of-period wealth. How then is the decision made? According to the widely accepted rationality axioms of von Neumann and Morgenstern (1947) a rational agent behaves as if maximizing expected utility: with the aid of a suitably chosen, monotonically increasing, utility function, the probability distributions of end-of-period wealth are first transformed into probability distributions of utility and then, from these distributions, the one with the highest mean is selected.

In a large body of economic literature the expected-utility rule has proved to be a flexible tool for modeling decision making under risk.

* This paper is a concise presentation of some of the ideas spelled out in a much broader context in Sinn (1980).

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Unfortunately however, flexibility is not necessarily an advantage. It may as well be a sign of a lack of content. Indeed, as long as nothing is known about the utility function except that it is a strictly increasing, and perhaps concave, function, the behavioural implications of the expected-utility rule are quite vague.

This article is therefore devoted to the task of establishing a hypothesis on the shape of the utility function for both a myopic decision maker and a multiperiod optimizer. To do this, use will be made of psychophysical laws which, up to now, have received little attention in economics.

The psychophysical laws are reviewed in an interpretative way in section 2. On their basis, in section 3 the preference hypothesis itself is formulated. Section 4 offers a few comments on competing proposals regarding the shape of the utility function that have been made in the literature.

2. The relevant psychophysical laws

2.1. The sensation function

There are two psychophysical laws needed to establish the preference hypothesis. Our discussion starts with the law describing the functional relationship between a stimulus intensity r and its subjective magnitude s:

$$s = s(r). \tag{1}$$

An early hypothesis concerning the shape of this 'sensation function' is Fechner's (1860, 1877, 1888) logarithmic law, saying that

$$s = a + b \ln r, \tag{2}$$

where a is arbitrary since it depends on the unit [1] of r, but b is characteristic for a special stimulus continuum. Fechner referred to Weber's (1834, 1846) threshold experiments which indicated that the

[1] When a new unit is chosen which is $1/x$ times as large as the old one, (2) becomes $s = (a + b \ln x) + b \ln r$.

smallest just noticeable increment to a stimulus intensity [2] is, within the relevant range, a constant fraction of the intensity level itself [3]. By assuming that, regardless of the intensity level, all just noticeable increments are subjectively equal, he was able to show that this remarkable constancy implies the logarithmic sensation function (2). Unfortunately, however, Fechner did not provide a legitimation for his assumption. Thus it is not surprising that his law has often been rejected.

Another hypothesis which has been very popular in recent years is Stevens' (1975) power law

$$s = \alpha r^{\Theta}, \quad \alpha > 0, \Theta > 0. \tag{3}$$

Here α is meaningless for it depends on the unit of r, but Θ characterizes the stimulus type [4]. Stevens derived his law purely inductively from a great many experiments carried out at Harvard Laboratory of Psychophysics. In these experiments people were asked to estimate stimulus intensities by the direct use of real number scales. Surprisingly, for a given continuum, the *number-matching* estimates turned out to be a power function of the stimulus intensities. Examples for the continua considered are loudness, vibration, lightness, length of straight lines, saturation of colour mixtures, salt concentration and heaviness.

Although there cannot be any doubt concerning the validity of Stevens' measurements, it is unclear whether the numbers people chose really did measure the subjective magnitude or sensation of the objective stimulus intensities being presented. As Garner et al. (1956: 155–157), Attneave (1962: 623–627) and Ekman (1964) have rightly pointed out, such interpretation requires a strong assumption: namely that the subjective magnitudes of numbers are identical with their objective ones. Since this assumption is as arbitrary as Fechner's assumption that all just noticeable increments are subjectively equal, we

[2] The stimulus must be measurable by a ratio scale.

[3] For some time it was popular to deny Weber's law (see for instance Boring (1942: 138 f.)) since the fraction increases for very high and very low stimulus intensities. However, as Stevens (1951: 35) notes, the range where the fraction is indeed constant, covers 99.9% (!) of the practically appearing stimulus intensities.

[4] When a new unit is chosen which is $1/x$ times as large as the old one, (3) becomes $s = (x^{\Theta}\alpha)r^{\Theta}$.

must conclude that number matching is in fact a *cross-modality matching*, where number sensation is set equal to the sensation of another kind of stimulus. Consequently, all we derive from Stevens' experiments is a system of relationships between the underlying true sensation functions, which themselves are unknown.

A variety of shapes of these true sensation functions is compatible with Stevens' empirical findings. The functions may be of Stevens' power type [5]; yet, as shown by Ekman (1964), they may just as well be of Fechner's logarithmic type [6]. Note, though, that they cannot be partly of one and partly of the other type. Whenever the true sensation function for a particular continuum is found to belong to one of the two types of functions, then, provided Stevens' empirical results are valid, the true sensation functions for *all* other continua of necessity have to belong to the same type. The reason is that a number-matching experiment of the kind

$$\alpha_1 + \Theta_1 \ln r_1 = \alpha_2 r_2^{\Theta_2}, \tag{4}$$

$$1 \equiv a,\ 2 \equiv n \text{ or } 1 \equiv n,\ 2 \equiv a,$$

would never yield the power relation Stevens observed.

Fortunately a method exists for determining the family to which the still unknown sensation functions belong – the method of interval or category estimation. Here the experimental subject is asked to classify given stimuli into equidistant magnitude categories or to manipulate a set of stimulus intensities so that the distances between them seem to be subjectively equal. The basic difference between interval estimation and the number-matching methods of measuring employed by Stevens is that, instead of comparing the number continuum with another continuum, the experimental subject is concerned with only one con-

[5] Suppose that the true sensation function for numbers is $s_n = \alpha_n r_n^{\Theta_n}$ and that for another continuum the sensation function is $s_a = \alpha_a r_a^{\Theta_a}$. Then it follows from $s_n = s_a$, i.e., $\alpha_n r_n^{\Theta_n} = \alpha_a r_a^{\Theta_a}$, that a number-matching experiment will yield a power function with the exponent Θ_a / Θ_n:

$$r_n = \left(\frac{\alpha_a}{\alpha_n}\right)^{1/\Theta_n} r_a^{\Theta_a/\Theta_n}.$$

[6] Let $s_n = \alpha_n + \Theta_n \ln r_n$ and $s_a = \alpha_a + \Theta_a \ln r_a$. Then, since in this case number matching means that $\alpha_n + \Theta_n \ln r_n = \alpha_a + \Theta_a \ln r_a$ or, equivalently, $r_n = e^{(\alpha_a - \alpha_n)/\Theta_n} r_a^{\Theta_a/\Theta_n}$, a power function observed in number-matching experiments is compatible with true sensation functions obeying Fechner's law.

tinuum: for a *given* kind of stimulus, the task is to compare an increase in stimulus intensity on a certain level with an increase in the intensity on another level. Hence the experimental subject is asked for precisely the piece of information needed to find out about the type of his sensation function.

In order to be able to compare the results obtained by the method of interval estimation, define

$$\eta \equiv -\frac{s''(r)}{s'(r)}r. \tag{5}$$

The parameter η is the negative elasticity of marginal sensation $s'(r)$ i.e., $\eta \equiv -[\partial s'(r)/\partial r][r/s'(r)]$. It measures the degree of curvature of the sensation function $s(r)$ at point r. A value of $\eta = 0$ characterizes a linear sensation function, a value of $\eta > 0$ a concave function and a value of $\eta < 0$ a convex function. In the special cases of the power and the logarithmic functions, regardless of r, the measure η takes on a constant value and may hence be utilized to indicate which class prevails. It can easily be calculated that

$$\eta = \begin{cases} 1 - \Theta & \text{if } s = \alpha r^{\Theta}; \quad \alpha, \quad \Theta > 0; \quad \text{(Stevens' law)}, \\ 1 & \text{if } s = \alpha + \Theta \ln \quad r; \quad \Theta > 0; \quad \text{(Fechner's law)}. \end{cases} \tag{6}$$

If $\eta < 1$, so that the sensation function is only moderately concave, or even linear or convex, then Stevens' law prevails. In the special case $\eta = 1$, where the sensation function is more curved than under Stevens' law, Fechner's law shows up. If $\eta > 1$, there is a still stronger curvature that, strictly speaking, excludes both Stevens' and Fechner's law, but is obviously nearer to the latter than to the former.

The first interval-method result was obtained by Plateau (1872). He asked painters to mix a grey color, so that its lightness was halfway between black and white, and found $\eta = 2/3$. However, Plateau was soon corrected by Delboef (1873: esp. 50–101), who asked the experimental subjects to produce the grey by changing the ratio of black and white areas on a rotating disk. This more exact method yielded $\eta = 1$. Repeating Delboef's experiment, Guilford (1954: 199–200) derived a sensation function which was even a little more curved. The value

$\eta = 1.15$ can be calculated from his tables [7]. Later even Stevens and Galanter (1957) and Stevens (1961) corroborated the tendency of these results. They admitted that, in comparison to number matching, the sensation functions observed in interval experiments show a systematic bias towards higher values of η, i.e., a bias towards Fechner's law. In the sequel there have been a lot of further investigations [8]. The main result, which is a triumphal rehabilitation of Fechner's law, is summarized in a review article by Ekman and Sjöberg (1965: 464): 'The logarithmic relation between indirect interval and direct ratio [number matching] scales is now a well-established fact for a great number of continua.' Since, however, for *all* of Stevens' continua, including the number continuum [9], the true sensation functions must be logarithmic if a logarithmic function is shown for only *one* continuum, we have thus come to the first of the psychophysical laws we need for establishing our preference hypothesis – Fechner's law [10].

2.2. *The relativity law*

The second law we need is Weber's (1834: 161, 172–173) relativity

[7] Suppose the sensation function is of the general form $s = \alpha + \beta\Theta r^{\Theta}$, $\beta > 0$, $\Theta \neq 0$. Then for two stimulus intensities r_1 and r_2 and their psychological mean $\dot{r}$ we have the relation $\alpha + \beta\Theta \dot{r}^{\Theta} = (\alpha + \beta\Theta r_1^{\Theta} + \alpha + \beta\Theta r_2^{\Theta})/2$ or equivalently $\dot{r}^{\Theta} = (r_1^{\Theta} + r_2^{\Theta})/2$. Given Guilford's results $r_1 = 100$, $r_2 = 2500$, and $\dot{r} = 411$ it is possible to calculate, by an iterative procedure, $\Theta = 1 - \eta = -0.1529$.

[8] Compare especially the investigations of Galanter and Messick (1961) and Eisler (1962), who demonstrate the logarithmic relation for the loudness continuum. These two studies are quite important since it is the loudness continuum which had served as a reference basis in many cross-modality experiments made by Stevens (see, e.g., 1966, 1975) in order to check the number-matching estimations.

[9] The fact that a number system is chosen, where the difference between the lengths of two written numbers is equal to the difference between the logarithms of their values, is in line with this result.

[10] A strong indication for the validity of Fechner's law can also be found in neurophysiological measurements, if we consider that the intensity of a simple physical stimulus is transformed into an intensity of electric current in the receptor organ, which itself determines the impulse frequency of the corresponding nerve fibre. A first result stems from Fröhlich (1921: esp. 15). For the luminosity sensation he found the intensity of current to be a logarithmic function of light intensity. Later Hartline and Graham (1932, 1938), Fuortes (1959), and Fuortes and Poggio (1963) similarly discovered that the light intensity is transformed into impulse frequency according to a logarithmic function. Galambos and Davis (1943) and Tasaki (1954) found corresponding results for the loudness continuum. Of course compatibility with Fröhlich's result then requires that the impulse frequency be proportional to the intensity of current in the receptor organ. That this is indeed the case was shown by Katz (1950) and Fuortes and Poggio (1963).

law [11], which is closely related to the first law but not identical to it. Weber derived it as a generalization of his threshold experiments. However, it is also a generalization of Fechner's law and is even compatible with Stevens' law [12]. Weber's law refers to the same stimuli as these laws and states that our senses are concerned not with absolute but with relative stimulus changes. Equal relative stimulus changes seem equally important, equally intensive, and are perceived as being equal: they are interpreted as the same information.

We thus detect an object under strong or weak light, because the light intensity ratios on the retina stay constant, and also independently of its distance, since not the absolute magnitude of the retina picture but its proportions matter. We perceive a melody independently of the octave in which it is played, for the frequency ratios stay constant [13], and independently of the musician's distance, as only the loudness ratios matter. Our sensory system has no difficulty in steering our car through daily traffic although during its evolutionary genesis it had learned merely to command our comparatively poorly equipped natural body, and we live our luxurious lives as self-evidently as our ancestors lived under much more modest circumstances. Moreover, how would Niels Bohr have been able to explain the atom structure by a planetary model if he had not thought in terms of magnitude ratios?

The reason for the phenomenon of ratio perception seems to be that the information we receive from our environment is encoded in a ratio language: equal loudness ratios, equal light intensity ratios or equal magnitude ratios do indeed indicate equal pieces of information. Thus it is not surprising that in the long run of evolution our sensory system has learned to decode these pieces of information economically, namely by neglecting the information about the absolute intensities and instead concentrating on their ratios. We should accept this speciality of our

[11] In Weber's (1834: 172) own words the law is: 'In observando discrimine rerum inter se comparatarum non differentiam rerum, sed rationem differentiae ad magnitudinem rerum inter se comparatarum percipimus.' This general relativity law was also proposed by Wundt (1863: esp. 65–70), Grotenfelt (1888), Meinong (1896), Lipps (1902; 1905: 231–287) and others. Recently a formal axiomatization of a 'relation theory' was provided by Krantz (1972).

[12] See Stevens (1975: 18).

[13] The fact that a step smaller than an octave changes the tonality does not contradict the hypothesis of ratio perception since the frequency ratios change in this case. If a melody is played in A-major it sounds different from that in C-major, for example. The reason is J.S. Bach's equal temperament of the scale, which made the ratios of all frequency pairs integers in order to avoid tremolo.

perception apparatus as a matter of fact and ask only what it implies for the shape of the von Neumann-Morgenstern utility function.

3. The risk preference function

This section establishes a hypothesis in the shape of the von Neumann-Morgenstern function for both the myopic decision maker and the multiperiod optimizer. For the former only Weber's law is required, whereas for the latter reference will also be made to Fechner's law.

3.1. The myopic decision maker

We start with the myopic model, assuming that the decision maker has to evaluate probability distributions of actual wealth. Wealth is either assumed to serve as a means of satisfaction as such or is regarded as an unspecified quantity index of all future consumption possibilities. Taking into account Weber's relativity law, we formulate the following

Weber axiom. Equal relative changes in wealth are perceived as equally significant.

This axiom provides the psychophysical basis of our hypothesis. It yields a true description of reality if wealth perception runs parallel to the perception of other psychophysical stimuli like, for instance, the 'length of a straight line' or the 'magnitude of a real number'.

In order to find out what the axiom implies for the shape of the utility function, consider Barrois' (1834: esp. 260–261) problem of determining the maximum willingness to pay for a full-coverage insurance contract. Let $U(\cdot)$, $U' > 0$, denote the decision maker's von Neumann-Morgenstern utility function, let a denote the initial wealth and let C, $C \geq 0$, denote a random variable characterizing the loss distribution the decision maker faces for the period in question. In the absence of insurance, end-of-period wealth is $a - C$. The corresponding level of expected utility is $\mathrm{E}[U(a - C)]$ where $\mathrm{E}(\cdot)$ is the expectation (or: mean-value) operator. Suppose now, insurance is bought at a premium p. Then, end-of-period wealth takes on the non-random value $a - p$ and hence brings about the utility $U(a - p)$. The maximum insurance premium the decision maker is willing to pay, p_{max}, is determined in such a way that he or she is just indifferent between buying insurance and staying uninsured:

$$\mathrm{E}[U(a-C)] = U(a-p_{\max}). \tag{7}$$

Applying the inverse function of U, U^{-1}, to both sides of this equation we have

$$p_{\max} = a - U^{-1}\{\mathrm{E}[U(a-C)]\}. \tag{8}$$

The expression $U^{-1}\{\mathrm{E}[U(W)]\}$ is usually called the 'certainty equivalent' of a probability distribution W of end-of-period wealth. The certainty equivalent gives the non-random level of wealth which the decision maker considers equivalent with the probability distribution W. Quite plausibly, therefore, (8) says that the decision maker is at most willing to pay an insurance premium amounting to the difference between initial wealth and the certainty equivalent of the end-of-period wealth distribution in the absence of insurance.

Now suppose the initial wealth and also the loss distribution are changed, although without altering the distribution of the loss-wealth ratio C/a. Then, according to the Weber axiom, the nature of the decision problem is unchanged and consequently the maximum relative part of wealth the decision maker is willing to pay for the insurance contract should not change either. In other words, multiplying $p_{\max}$, a and C by a factor λ, $0 < \lambda \neq 1$, we should find the equation

$$\lambda p_{\max} = \lambda a - U^{-1}\{\mathrm{E}[U(\lambda a - \lambda C)]\}, \tag{9}$$

which requires a linear homogeneity in the certainty equivalent. According to a theorem of Aczél (1966: 151–153), the only strictly increasing functions $U(\cdot)$ satisfying this property are $\ln w$ and γw^{γ}, $\gamma \neq 0$, and arbitrary linear transformations of them. We shall henceforth call these functions the *Weber functions*.

With the Weber functions, our argument establishes a hypothesis on the shape of the utility function which Pratt (1964) and Arrow (1965) classified as *constant relative (proportional) risk aversion* and which Pollak (1970) called weak homogeneity. Utilizing the Pratt-Arrow measure of relative risk aversion [14]

$$\varepsilon \equiv -\frac{U''(w)}{U'(w)}w, \tag{10}$$

where w is a particular variate of W, we can write the Weber functions as [15]

$$U(w)=\begin{cases}\ln w, & \varepsilon=1,\\ (1-\varepsilon)w^{1-\varepsilon}, & \varepsilon\neq 1.\end{cases} \tag{11}$$

Instead of employing Weber's relativity law it might be tempting to refer to Fechner's logarithmic sensation law in order to establish a hypothesis on the shape of the von Neumann-Morgenstern function $U(w)$. Suppose the subjective magnitude of wealth is identical with the subjective magnitude of the numbers by which wealth is measured, and suppose moreover the subjective magnitude of wealth can also be identified with its introspective utility. Then the introspective utility-of-wealth function $u(w)$ has to be logarithmic, i.e.,

$$u(w)=\ln w, \tag{12}$$

or any strictly increasing linear transformation thereof. Thus, at first glance, it seems that the von Neumann-Morgenstern function $U(w)$ is logarithmic, too. This, indeed, is the reasoning of Bernoulli (1738), who postulated a logarithmic utility-of-wealth function [16] (anticipating

[14] The parameter ε is formally analogous to η defined in (5) and hence has a similar meaning. It can be shown that, for a wealth distribution with a small variance $\sigma^2(W)$, the difference between the mean and the certainty equivalent is approximated by $\sigma^2(W)\varepsilon/[2E(W)]$. See, e.g., Pratt (1964).

[15] It can be shown that these functions depict the same preference structure as a homothetic indifference curve system in the μ-σ diagram, if, for (nearly) arbitrarily chosen distributions, σ/μ is small and consequently a quadratic approximation can be used, or if all distributions to be evaluated belong to the same linear class such that a distribution is completely described by two parameters only. (Here μ and σ denote the expected value and standard deviation of the wealth distribution.) See Sinn (1980: ch. III A 2.2.). This shows that our hypothesis was already implicitly anticipated by Hicks (1967: 114), who regarded homothetic indifference curves as the 'standard case', and by I. Fisher (1906: 408–409), who made the subjective weight of risk depend on the coefficient of variation, σ/μ.

[16] Bernoulli argued that a realistic utility-of-wealth function should have the property that equal relative changes in wealth bring about equal absolute changes in utility. Although today Bernoulli is famous for the expected-utility rule as such, it should not be overlooked that during the subjectivist discussion of the last century this argument for the logarithmic function was regarded as the central point of his essay. See the preface of L. Fick to the 1896 edition of the German translation of Bernoulli's article.

Fechner by more than a hundred years) and proposed the rule max E[ln W]. However, it is well known that the introspective utility function should not be identified with the von Neumann-Morgenstern utility function, for there is no reason for two persons with the same utility function for non-random wealth acting alike if probability distributions of wealth are to be evaluated [17].

By referring to Weber's law we have avoided Bernoulli's error. Thus it is natural that in (11) the logarithmic function turned out to be only one of the possibilities. In order to clarify the difference between the Weber functions (11) and the subjective utility function (12), let us follow a proposition of Krelle (1968: 144–147). Krelle suggested that von Neumann-Morgenstern utility should be determined in two steps. In the first step a variate of wealth w is transformed into subjective utility $u(w)$. In the second step this subjective utility is transformed into von Neumann-Morgenstern utility by means of a *specific risk preference* function $\Omega(u)$. Hence

$$U(w) = \Omega[u(w)]. \tag{13}$$

The shapes of $U(w)$ and $u(w)$ are determined by (11) and (12). Given these shapes, the only admissible versions of the specific risk preference functions are [18]

$$\Omega(u) = \begin{cases} u, & \varepsilon = 1, \\ (1-\varepsilon)e^{(1-\varepsilon)u}, & \varepsilon \neq 1, \end{cases} \tag{14}$$

or strictly increasing linear transformations thereof.

As is well-known the curvature of the von Neumann-Morgenstern function, which is measured by ε from (10), determines the individual's

[17] Consider the vehement criticism of Allais (1952, 1953) and the clarifying paper of Baumol (1958).

[18] In general, for $u(w) = \alpha + \Theta \ln w$, $\Theta > 0$, (14) becomes

$$\Omega[u(w)] = \begin{cases} \alpha + \Theta \ln w, & \varepsilon = 1, \\ (1-\varepsilon)e^{(1-\varepsilon)(\alpha + \Theta \ln w)} = e^{(1-\varepsilon)\alpha}(1-\varepsilon)w^{\Theta(1-\varepsilon)}, & \varepsilon \neq 1. \end{cases}$$

Since the von Neumann-Morgenstern function $U(\cdot)$ is defined up to a strictly positive linear transformation, obviously it is only the factor Θ which has a behavioural implication in the case $\varepsilon \neq 1$, given the value of ε. However, since the whole set of possible von Neumann-Morgenstern functions is independent of Θ, we arbitrarily set $\Theta = 1$. The degree of risk aversion is then only modeled by ε.

risk aversion in evaluating probability distributions of wealth. According to Krelle's hypothesis (13), this degree of risk aversion is traced back to both the curvature of $U(\cdot)$ and that of $\Omega(\cdot)$. The reader may easily check that the subjective utility function (12) by itself implies $\varepsilon = 1$, that is, some positive degree of risk aversion. It is possible that the specific risk aversion function is linear so that this degree of risk aversion is not modified. This is Bernoulli's case which is depicted by the first line on the right-hand side of (14). In general, however, as captured by the second line in (14) the specific risk preference function is curved and will hence exhibit a significant influence on the evaluation of risks.

In the general case the specific risk preference function takes on the particular mathematical form of the exponential function. This function was first suggested by Freund (1956) for the objective continuum and was shown by Pfanzagl (1959a: 39–41, 55–57; 1959b: 288–292) and Pratt (1964: 130) to be characterized by what, in the Pratt-Arrow terminology, is called *constant absolute risk aversion*. Hence our preference hypothesis not only implies constant relative risk aversion on the objective (wealth) continuum, but also constant absolute risk aversion on the subjective (utility) continuum.

In line with the Pratt-Arrow definitions, the degree of absolute risk aversion of the specific risk preference function is $-\Omega''(u)/\Omega'(u)$. Calculating this measure for (14) we find

$$\varepsilon = 1 + \left(-\frac{\Omega''(u)}{\Omega'(u)} \right). \tag{15}$$

Thus, the degree of relative risk aversion (ε) on the objective continuum is one plus the degree of absolute risk aversion ($-\Omega''/\Omega'$) on the subjective continuum. If the degree of absolute risk aversion on the subjective continuum is strictly positive, that is, if $\Omega(\cdot)$ is concave, then the risk aversion predetermined by the logarithmic subjective utility function (12) is reinforced: relative risk aversion on the objective continuum is above unity. If, on the other hand, the degree of absolute risk aversion on the subjective continuum is strictly negative, that is, if $\Omega(\cdot)$ is convex and risk loving on the subjective continuum prevails, then the risk aversion predetermined by the logarithmic subjective utility function (12) is weakened: relative risk aversion on the objective continuum is less than unity.

3.2. The multiperiod optimizer

It is unclear whether the myopic or the multistage optimizing model is a better description of reality, but there is no question that the latter is the better normative approach. Thus we attempted to show how a multiperiod optimizer should behave if his or her preferences were compatible with the psychophysical laws.

For this task our basic assumption is that the multiperiod preference functional can be written in general as [19]

$$\mathrm{E}\left\{\psi\left[\sum_{t=0}^{T} f(C_t)\lambda_t\right]\right\}; \quad \psi'(\cdot)>0,\ f'(\cdot)>0,\ \lambda_t>0\forall t. \tag{16}$$

Here $\sum_{t=0}^{T} f(C_t)\lambda_t$ is the deterministic multiperiod preference functional, where T is the planning horizon, C_t consumption in period t, C_T final wealth, $f(\cdot)$ the felicity-of-consumption function, and λ_t the felicity discount factor. In the version $T=\infty$ and $\lambda_t=\lambda^t$, $0<\lambda<1$, this preference functional was provided an axiomatic basis by Koopmans (1960). It is well known that Koopmans' separability assumption responsible for the additivity is crucial. However, we follow common usage and accept it as a simplification [20]. $\psi(\cdot)$ is a specific risk preference function which is introduced to allow for degrees of risk aversion other than that implied by the curvature of $f(\cdot)$, the genuine task of which is to model the decision maker's intertemporal preferences rather than his risk preferences. Our aim is now to specify this function $\psi(\cdot)$ and the felicity function $f(\cdot)$ by reference to the psychophysical laws described above.

The first piece of information can be derived from the previous discussion concerning the myopic case. Suppose initial wealth w is replaced by a factor x measuring the level of a (deterministic) consumption path $(xc_0^*, xc_1^*, \ldots, xc_T^*)$ which is just the multiple of a standard path $(c_0^*, c_1^*, \ldots, c_T^*)$. Then, according to the Weber axiom, (16) should have the property that the decision maker, in evaluating

[19] For an axiomatic foundation of this preference functional see Sinn (1980: ch. IV B 2.1.1).

[20] The additivity can be rationalized if the complementarities between consumption levels in the different periods are the weaker the greater the distances between these periods, for then their disturbing character can be reduced by a simple lengthening of periods. This has been pointed out by Arrow and Kurz (1970: 11–12).

gambles on x, behaves as if following the rule max $\mathrm{E}[U(X)]$, where $U(\cdot)$ exhibits constant relative risk aversion. Thus, for precisely the same reason as that bringing about (11), we should have

$$\psi\left[\sum_{t=0}^{T} f(xc_t^*)\lambda_t\right] = \begin{cases} \ln x + \psi\left[\sum_{t=0}^{T} f(c_t^*)\lambda_t\right], & \varepsilon' = 1, \\ x^{1-\varepsilon'}\psi\left[\sum_{t=0}^{T} f(c_t^*)\lambda_t\right], & \varepsilon' \neq 1, \end{cases} \tag{17}$$

where ε' is the Pratt-Arrow measure of relative risk aversion characterizing the shape of $U(\cdot)$.

Unfortunately this information is insufficient to determine both $\psi(\cdot)$ and $f(\cdot)$, for it refers to a sum effect of both functions. What we need is further information concerning the shape of either $\psi(\cdot)$ or $f(\cdot)$. For the latter such information is indeed available. Assume that the felicity of consumption can be identified with the subjective magnitude of consumption and assume that the latter is determined by the subjective magnitude of the number by which consumption is measured. Then the felicity function obeys Fechner's law. A possible objection to this outcome is that the subjective magnitude of one period's consumption cannot necessarily be determined independently of consumption in other periods. However, this possibility has already been excluded with the assumption of the separable Koopmans-preference functional. Thus we can formulate the following

Fechner axiom: *Equal relative changes of a period's consumption bring about equal absolute changes of felicity in that period.*

Obviously the preference functional for deterministic multiperiod planning is then

$$\sum_{t=0}^{T} \lambda_t \ln c_t, \tag{18}$$

where we omit an additive constant and assume [21] $\sum_{t=0}^{T}\lambda_t = 1$. This

[21] As in the myopic case this assumption has no behavioural implications and is made for simplicity only. Compare fn. 18.

preference functional was already mentioned by Modigliani and Brumberg (1955: 396, fn. 15) in an allusion to psychophysics. It is in line with the logarithmic utility-of-wealth function (12) if consumption in each period is proportional to initial wealth, i.e., if wealth is indeed an adequate quantity index of future consumption possibilities, as we had assumed in the myopic model.

Now it is easy to find the specific risk preference function $\psi(\cdot)$ compatible with both (17) and (18). It is again the function (14) from the myopic case, for combining $\psi(\cdot)=\Omega(\cdot)$ and (18) is the only way to satisfy (17). Accordingly, the possibilities for the preference functional (16) turn out to be

$$\mathrm{E}\left\{\psi\left[\sum_{t=0}^{T} f(C_t)\lambda_t\right]\right\} = \begin{cases} E\left(\sum_{t=0}^{T}\lambda_t \ln C_t\right), & \varepsilon'=1, \\ (1-\varepsilon')\mathrm{E}\left(\prod_{T=0}^{T} C_t^{(1-\varepsilon')\lambda_t}\right), & \varepsilon'\neq 1, \end{cases} \tag{19}$$

where the latter is derived from

$$\mathrm{E}\left\{(1-\varepsilon')\mathrm{e}^{(1-\varepsilon')(\sum_{t=0}^{T}\lambda_t \ln C_t)}\right\}.$$

For a world of stochastic constant returns to scale and under the assumption $\lambda_t=\lambda^t$, $0<\lambda<1$, the implications of these preference functionals have already been studied by Pye (1972). He showed that a myopic utility-of-wealth function relevant for the actual decision can be derived which is characterized by constant relative risk aversion, the degree of which changes with time. Specifically, if ε_t is the Pratt-Arrow measure of relative risk aversion relevant for the probability distribution of wealth appearing at the time t and depending on the decision at $t-1$, then

$$\varepsilon_t = 1-(1-\varepsilon')\sum_{\tau=t}^{T}\lambda_\tau. \tag{20}$$

Furthermore he found that the part of wealth at time t reserved for consumption during the following period is given by

$$\alpha_t = 1 \Bigg/ \left(1 + \frac{\sum_{\tau=t+1}^{T} \lambda_\tau}{\lambda_t}\right). \tag{21}$$

Verbally the rule (20) means that relative risk aversion nears unity as time goes by. Thus with $\varepsilon' < 1$ it is possible to depict the everyday observation that risk aversion increases with age.

In a study of another family of preference functionals Samuelson (1969) admitted his surprise in finding a time *in*dependence of risk aversion, for he had expected risk aversion to increase with age, arguing that the 'chance to recoup' is the greater the younger a person is. One might claim that this supposition is now rationalized by our preference hypothesis. However, this interpretation is incorrect, for, contrary to its intention, decisions taken in the youth are of greater importance than those taken in late years. On the one hand, according to (21), consumption is always proportional to wealth; on the other hand, constant relative risk aversion means that the relative wealth distribution (e.g. determined by the portfolio structure) is chosen independently of the absolute wealth level. Therefore, in comparison to what the situation would otherwise have been, a given percentage change in wealth at any point in time causes an equal percentage change in consumption in each of the following periods up to the horizon, including an equal percentage change in final wealth. This implies that the younger a person is, the more lifetime utility (18) is affected and that, even for a very young person, there is no chance to recoup at all. Since according to (16) distributions of lifetime utility are evaluated after applying $\psi(\cdot)$ as a specific risk preference function, the correct interpretation of the time dependence of risk aversion is now obvious. For old persons the dispersion of lifetime utility is so small that the curvature of $\psi(\cdot)$ can be neglected, so that risk neutrality on the subjective continuum is roughly the appropriate attitude. This is equivalent to saying that, for evaluating probability distributions of wealth, the relative risk aversion of old persons should be near unity. However for young persons with great dispersion of lifetime utility the curvature of $\psi(\cdot)$ generally cannot be neglected. Thus, for them, risk preference or risk aversion on the subjective continuum plays an important role in finding the optimal decision, so that relative risk aversion may differ quite substantially from unity.

According to (21) our preference hypothesis implies furthermore that the propensity to consume out of wealth depends on the decision maker's time preference and the distance of the time horizon, but not on present or future investment opportunities. The reason is that the income and the substitution effect of a change in expected returns just offset each other. This simplifies the actual decision enormously, for it is neither necessary for the decision maker to chose the actual risk projects simultaneously with the consumption level nor to know which investment opportunities will be available in future. In sum, in order to derive the optimal decision at any point in time the multiperiod optimizer needs only to know his or her *own* preferences and the *current* opportunity set of risk projects.

With this result the laws of Weber and Fechner provide to a certain extent a rehabilitation of the simplest risk theoretic model of the expected utility-of-wealth maximizer. This is at first glance surprising, but in fact not difficult to explain. On the one hand, the phenomenon of ratio perception seemed to originate from the evolutionary optimization process which adapted our organism to the ratio code in which environmental signals are written. On the other hand, with the assumption of stochastic returns to scale we made the economic decision maker operate in a world where the relevant information is also formulated in a ratio code. Is it then still surprising that he finds simple behavioural rules for this world as well?

4. Competing approaches

Our psychophysical hypothesis on the shape of the von Neumann-Morgenstern function $U(w)$ is not the only one which has been established. There are others which contrast sharply with it.

Those of Törnqvist (1945), Friedman and Savage (1948) and Markowitz (1952) should be mentioned first. These authors have in common their construction of $U(\cdot)$ from concave and convex segments so that the preference can be depicted not only for insurance contracts but also for gambling. Yet their approach is not very satisfactory, for gambling contradicts the expected utility rule as such. On the one hand, contrary to the fundamental axiom of ordering, gambling normally implies that the decision maker is not only interested in the eventual probability distributions, but also in the way they are generated. On the other hand, gamblers frequently derive their decisions from some mystic rules which are not compatible with the other rationality axioms underlying the expected-utility rule either. An argument related to the latter is expressed very clearly by Hicks (1962: 793), who states: '...gambling is relaxation. To expect consistency in gambling is futile for gambling is a rest from consistency.'

Another hypothesis, established by Arrow (1965: 28–44, 1970: 90–120), is that relative risk aversion increases with wealth. Arrow's argument is twofold, theoretical and empirical. Concerning the former, he is able to show that there is an axiom system from which not only the expected-utility rule itself, but also a utility boundedness theorem stating that $\lim_{w \to \infty} U(w) < \infty$ and $\lim_{w \to 0} U(w) > -\infty$ can be derived. Then, since boundedness from below implies that relative risk aversion (ε) falls short of unity for $w \to 0$ and boundedness from above implies that it exceeds unity for $w \to \infty$, he concludes that relative risk aversion is increasing. It is not intended to discuss here the question of whether Arrow's axiom system or an alternative system, which does not imply the utility boundedness theorem, is the more realistic [22]. For, as Stiglitz (1969) points out, even if the utility function were bounded at its extremes it might have any shape in between. In particular suppose

$$U(w) = \begin{cases} w^{1-\varepsilon}, & w \geqslant \bar{w}, \\ \bar{w}^{1-\varepsilon}, & w \leqslant \bar{w}, \end{cases} \qquad \bar{w} \equiv \text{world's value},$$

where $\varepsilon < 1$, such that risk aversion increases with age. Then the utility function is bounded, but nevertheless cannot have behavioural implications different from those of the unbounded function $U(w) = w^{1-\varepsilon}$, $\varepsilon < 1$.

Arrow's empirical argument is that historically the stock of money has grown more quickly than wealth, which, on a basis of a portfolio model with a risky asset and money as the only safe asset, seems to imply that relative risk aversion increases with wealth. This argument is not convincing either. Firstly, the empirical investigations quoted by Arrow do not clearly support his hypothesis. The fact that the stock of real balances grew more quickly than wealth seems to be due to the

[22] A comparative discussion is given in Sinn (1980: ch. III C 2.).

secularly fallen interest rates rather than to the shape of the risk preference function. This indeed is the result of Latané (1960) and Meltzer (1963), who, in contrast to the other authors mentioned by Arrow, install interest rates as explanatory variables in their tests and come to the conclusion that the hypothesis of a unitary *partial* wealth elasticity of money cannot be rejected. Secondly, as Stiglitz (1969), Shell (1972) and others remarked, money demand as such cannot easily be explained by a stochastic portfolio model, for in reality there are short-run interest-bearing assets available which clearly dominate money. Thirdly, even if all this were not true, the secularly risen average age of population can easily explain Arrow's observation if a quite modest degree of risk aversion ($0 < \varepsilon < 1$) is the standard case, for then relative risk aversion increases with age, even though it does not increase with wealth.

5. Conclusion

In this article two psychophysical laws were presented in order to study their implications for risk preference functions. One is Weber's relativity law and the other Fechner's law, which was shown to be the only possibility that made the number-matching and interval-method measurements of psychophysics compatible. For the myopic expected utility-of-wealth maximizer we referred to Weber's law alone and derived the hypothesis of constant relative risk aversion. However, for the multiperiod optimizer, Fechner's law was needed as well. Together with a modified version of Koopmans' preference functional these two laws imply a family of preference functionals which are either of an additive logarithmic or a multiplicative Cobb-Douglas type. This family has very appealing implications in a world of stochastic constant returns to scale. For his actual decision the multiperiod optimizer exhibits constant relative risk aversion as the myopic optimizer does. However, the degree of this risk aversion generally changes over time: regardless of its initial value, with the passage of time it approaches the value of unity. Furthermore, the consumption decision is independent of actual and future investment opportunities. Thus the agent neither has to make the consumption decision simultaneously with the selection of an optimal risk project nor needs any information about the future except about his own preferences.

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